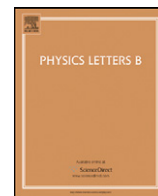


Contents lists available at [ScienceDirect](http://ScienceDirect)

## Physics Letters B

[www.elsevier.com/locate/physletb](http://www.elsevier.com/locate/physletb)OPE, charm-quark mass, and decay constants of  $D$  and  $D_s$  mesons from QCD sum rulesWolfgang Lucha<sup>a</sup>, Dmitri Melikhov<sup>a,b,c,\*</sup>, Silvano Simula<sup>d</sup><sup>a</sup> HEPHY, Austrian Academy of Sciences, Nikolsdorfergasse 18, A-1050 Vienna, Austria<sup>b</sup> SINP, Moscow State University, 119991 Moscow, Russia<sup>c</sup> Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria<sup>d</sup> INFN, Sezione di Roma III, Via della Vasca Navale 84, I-00146 Roma, Italy

## ARTICLE INFO

## Article history:

Received 31 January 2011

Received in revised form 11 May 2011

Accepted 16 May 2011

Available online 19 May 2011

Editor: B. Grinstein

## Keywords:

Nonperturbative QCD

QCD sum rules

Charmed mesons

## ABSTRACT

We present a sum-rule extraction of the decay constants of the charmed mesons  $D$  and  $D_s$  from the two-point correlator of pseudoscalar currents. First, we compare the perturbative expansion for the correlator and the decay constant performed in terms of the pole and the running  $\overline{MS}$  masses of the charm quark. The perturbative expansion in terms of the pole mass shows no signs of convergence whereas reorganizing this very expansion in terms of the  $\overline{MS}$  mass leads to a distinct hierarchy of the perturbative expansion. Furthermore, the decay constants extracted from the pole-mass correlator turn out to be considerably smaller than those obtained by means of the  $\overline{MS}$ -mass correlator. Second, making use of the OPE in terms of the  $\overline{MS}$  mass, we determine the decay constants of both  $D$  and  $D_s$  mesons with an emphasis on the uncertainties in these quantities related both to the input QCD parameters and to the limited accuracy of the method of sum rules.

© 2011 Elsevier B.V. Open access under [CC BY-NC-ND license](http://creativecommons.org/licenses/by-nc-nd/3.0/).

## 1. Introduction

The extraction of the decay constants of ground-state heavy pseudoscalar mesons within the method of QCD sum rules [1,2] poses a complicated problem. First, one derives an operator product expansion (OPE) for the correlation function

$$\Pi(p^2) = i \int d^4x e^{ipx} \langle 0 | T(j_5(x) j_5^\dagger(0)) | 0 \rangle \quad (1.1)$$

of two pseudoscalar heavy–light currents

$$j_5(x) = (m_Q + m) \bar{q}(x) i \gamma_5 Q(x). \quad (1.2)$$

Second, one considers the sum rule for this correlator. The sum rule is nothing but the expression of the fact that the representation of the Borelized correlator (1.1),  $\Pi(p^2) \rightarrow \Pi(\tau)$ , in the language of the intermediate hadron states is equal to the OPE for this correlator:

$$\Pi(\tau) = f_Q^2 M_Q^4 e^{-M_Q^2 \tau} + \int_{s_{\text{phys}}}^{\infty} ds e^{-s\tau} \rho_{\text{had}}(s)$$

$$= \int_{(m_Q+m)^2}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu). \quad (1.3)$$

Here,  $M_Q$  denotes the mass of the pseudoscalar meson  $P_Q$  containing the heavy quark  $Q$  while  $f_Q$  is its decay constant:

$$(m_Q + m) \langle 0 | \bar{q} i \gamma_5 Q | P_Q \rangle = f_Q M_Q^2. \quad (1.4)$$

For the correlator (1.1),  $s_{\text{phys}} = (M_{V_Q} + M_\pi)^2$  is the physical continuum threshold,  $M_{V_Q}$  being the mass of the vector meson containing  $Q$ . Obviously, for large values of  $\tau$  the contribution of the excited states decreases faster than the ground-state contribution and therefore  $\Pi(\tau)$  is dominated by the ground state.

The perturbative spectral density is obtained in the form of an expansion in terms of the strong coupling  $\alpha_s(\mu)$ :

$$\rho_{\text{pert}}(s, \mu) = \rho^{(0)}(s) + \frac{\alpha_s(\mu)}{\pi} \rho^{(1)}(s) + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \rho^{(2)}(s) + \dots \quad (1.5)$$

Clearly, the correlator (1.1) does not depend on the renormalization scale  $\mu$ ; however, both the perturbative expansion truncated at fixed order in  $\alpha_s$  and the truncated power corrections  $\Pi_{\text{power}}(\tau, \mu)$  given in terms of the condensates and the radiative corrections to the latter depend on  $\mu$ . Moreover, the relative magnitudes of the

\* Corresponding author at: HEPHY, Austrian Academy of Sciences, Nikolsdorfergasse 18, A-1050 Vienna, Austria.

E-mail address: [dmitri\\_melikhov@gmx.de](mailto:dmitri_melikhov@gmx.de) (D. Melikhov).

lowest-order contributions strongly depend on the choice of the renormalization scheme/scale.

Unfortunately, the truncated OPE allows one to calculate the correlator only at not sufficiently large  $\tau$ , such that the excited states give a sizable contribution to  $\Pi(\tau)$  in the corresponding  $\tau$ -range. In principle, the physical spectral density above the threshold might be measured experimentally; in practice, however, it is unknown. Therefore, one adopts the concept of duality to relate the contribution of the excited hadron states to the perturbative contribution: perturbative QCD spectral density  $\rho_{\text{pert}}(s)$  and hadron spectral density  $\rho_{\text{hadr}}(s)$  are close to each other at large values of  $s$ ; thus, for sufficiently large values of the parameter  $\bar{s}$ , (far) above the resonance region, one has the duality relation

$$\int_{\bar{s}}^{\infty} ds e^{-s\tau} \rho_{\text{hadr}}(s) = \int_{\bar{s}}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s). \quad (1.6)$$

In order to express the excited-state contribution by the perturbative contribution, we need to extend this relationship down to the value of the hadronic threshold  $s_{\text{phys}}$ . However, one has to be careful: the spectral densities  $\rho_{\text{pert}}(s)$  and  $\rho_{\text{hadr}}(s)$  are obviously different in the region near  $s_{\text{phys}}$ . Therefore, one finds

$$\int_{s_{\text{phys}}}^{\infty} ds e^{-s\tau} \rho_{\text{hadr}}(s) = \int_{s_{\text{eff}}(\tau)}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s), \quad (1.7)$$

where  $s_{\text{eff}}(\tau)$  is different from the physical threshold  $s_{\text{phys}}$ . A crucial (albeit rather obvious) observation is that, for the same reason which causes  $s_{\text{eff}}(\tau) \neq s_{\text{phys}}$ ,  $s_{\text{eff}}(\tau)$  has to be a function of the parameter  $\tau$  to render relation (1.7) exact.

By virtue of (1.7) we may rewrite the sum rule (1.3) as

$$\begin{aligned} f_Q^2 M_Q^4 e^{-M_Q^2 \tau} &= \int_{(m_Q+m)^2}^{s_{\text{eff}}(\tau)} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu) \\ &\equiv \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)). \end{aligned} \quad (1.8)$$

We refer to the right-hand side of this equation as the *dual correlator*. Evidently, even if the QCD inputs  $\rho_{\text{pert}}(s, \mu)$  and  $\Pi_{\text{power}}(\tau, \mu)$  are well known, the extraction of the decay constant requires a further criterion for fixing the effective continuum threshold  $s_{\text{eff}}(\tau)$ .

Noteworthy, Eq. (1.8) offers another way to convince oneself that  $s_{\text{eff}}(\tau)$  must be a function of  $\tau$ . In fact, the log slope on the left-hand side of (1.8) is independent of  $\tau$  and is equal to  $M_Q^2$  (which may be exactly known from experiment). Consequently, to guarantee the same  $\tau$ -behaviour on the right-hand side of (1.8), the effective threshold must be, in general, a function of  $\tau$ . In the literature the approximation of the threshold by some constant  $s_0$  independent of  $\tau$  is widely used. The corresponding dual correlator,  $\Pi_{\text{dual}}(\tau, s_0)$ , should therefore lead to the presence of a contamination of excited states on the left-hand side of Eq. (1.8). In principle, one may develop models for excited states in order to estimate (and subsequently remove) such a contamination. It is, however, clear that ultimately the same effect can be equivalently reached by considering an explicit  $\tau$ -dependence of the effective continuum threshold.

The *exact effective continuum threshold* – corresponding to exact values of the hadron mass and the decay constant on the left-hand side – is, of course, not known. Therefore, the actual extraction of hadron parameters from a sum rule consists in attempting (i) to find some reasonable approximation to the exact threshold and (ii) to control the accuracy of such an approximation. We stress again that the use of a  $\tau$ -dependent threshold is expected to improve the reliability of the extraction of the hadron parameter

considered compared with the *standard* procedure of assuming a constant,  $\tau$ -independent threshold.

Let us now look in detail at each step of the sum-rule calculation of the decay constant, starting with the OPE for the correlator.

## 2. OPE and heavy-quark mass

We use the perturbative spectral density  $\rho_{\text{pert}}(s)$  calculated in [3] to three-loop accuracy in terms of the pole mass of the heavy quark. The pole mass has been used in most of the sum-rule analyses since the pioneering work [2]. An alternative option is to reorganize the perturbative expansion in terms of the running  $\overline{\text{MS}}$  mass [4]. Since the correlator is known to  $\alpha_s^2$ -accuracy, the relationship between pole and  $\overline{\text{MS}}$  mass to the same accuracy is used. Explicit expressions for the perturbative spectral densities and power corrections may be found in [3,4] and are not given here.

Fig. 1 shows the perturbative spectral densities and the sum-rule estimates for  $f_D$  arising from (1.8) for our two choices of  $m_c$ : the pole mass  $m_{c,\text{pole}}$  and the running  $\overline{\text{MS}}$  mass  $\bar{m}_c(\mu)$ . The relevant OPE parameters are

$$\begin{aligned} \bar{m}_c(\bar{m}_c) &= (1.279 \pm 0.013) \text{ GeV}, \\ m(2 \text{ GeV}) &= (3.5 \pm 0.5) \text{ MeV}, \\ m_s(2 \text{ GeV}) &= (100 \pm 10) \text{ MeV}, \\ \alpha_s(M_Z) &= 0.1176 \pm 0.0020, \\ \langle \bar{q}q \rangle(2 \text{ GeV}) &= -(267 \pm 17) \text{ MeV}^3, \\ \langle \bar{s}s \rangle(2 \text{ GeV}) / \langle \bar{q}q \rangle(2 \text{ GeV}) &= 0.8 \pm 0.3, \\ \left\langle \frac{\alpha_s}{\pi} G G \right\rangle &= (0.024 \pm 0.012) \text{ GeV}^4. \end{aligned} \quad (2.1)$$

We employ a recent determination [5] of  $\bar{m}_c(\bar{m}_c)$ . The corresponding pole mass recalculated from the  $O(\alpha_s^2)$  relation between  $\bar{m}_c$  and  $m_{c,\text{pole}}$  is

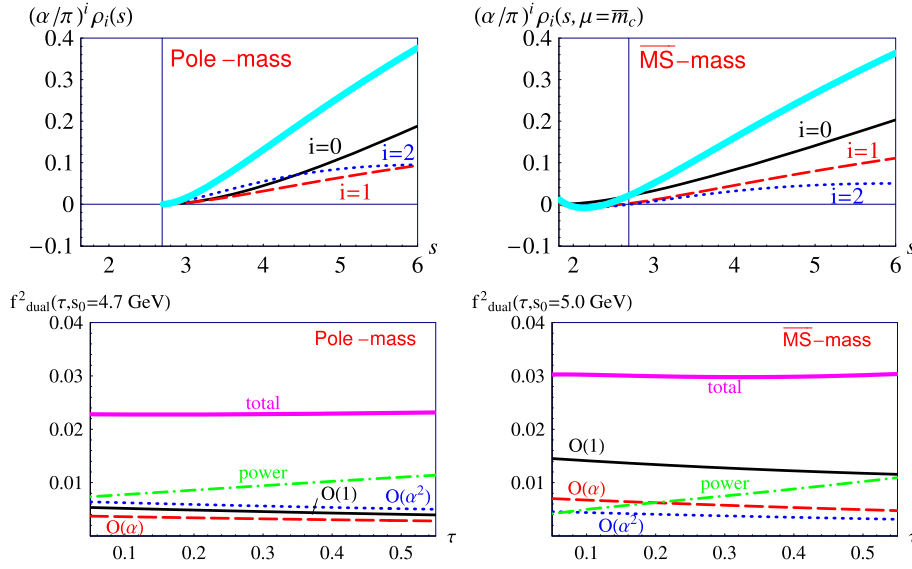
$$m_{c,\text{pole}} = 1.682 \text{ GeV}. \quad (2.2)$$

The sum-rule estimates shown in Fig. 1 are obtained for a  $\tau$ -independent effective threshold  $s_0$ . Its values, which prove to be different for the pole-mass OPE and the  $\overline{\text{MS}}$ -mass OPE, are found by requiring maximal stability of the extracted decay constant. Obviously, for heavy-light correlators and the resulting decay constants it makes a very big difference which precise scheme for the heavy-quark mass is employed.

Several lessons should be learnt from these plots:

- (i) The perturbative expansion for the decay constant in terms of the pole mass shows no signs of convergence; each of the terms – LO, NLO, NNLO – gives contributions of similar size. Therefore, there is no reason to expect higher orders to give smaller contributions.
- (ii) Reorganizing the perturbative series in terms of the  $\overline{\text{MS}}$  mass of the heavy quark leads to a visible hierarchy of the perturbative contributions.
- (iii) The absolute value of the decay constant extracted from the pole-mass OPE ( $f_D = 150 \text{ MeV}$ ) proves to be considerably smaller than that from the  $\overline{\text{MS}}$  scheme ( $f_D = 180 \text{ MeV}$ ).

Let us emphasize that, nevertheless, in both cases the decay constant exhibits perfect stability in a wide range of the Borel parameter  $\tau$ ! Thus we emphasize again that mere Borel stability is by



**Fig. 1.** OPE calculated in terms of the pole mass (left) and the  $\overline{\text{MS}}$  mass (right) of the  $c$  quark. First line: spectral densities; second line: corresponding sum-rule estimates for  $f_D$ . A constant effective continuum threshold  $s_0$  is fixed in each case separately by requiring “maximal stability” of the extracted decay constant. As the result,  $s_0$  turns out to be different in the two schemes. Bold line – total result, solid line (black) –  $O(1)$  contribution; dashed line (red) –  $O(\alpha_s)$  contribution; dotted line (blue) –  $O(\alpha_s^2)$  contribution; dash-dotted line (green) – power contributions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

far not sufficient to guarantee the reliability of the sum-rule extraction of bound-state parameters. We have already observed this feature in several examples in quantum mechanics [6].

Because of the obvious problems with the pole-mass OPE for the correlator, we shall make use of the OPE in terms of the running  $\overline{\text{MS}}$  mass for our extraction of the decay constants. Hereafter, the quark masses  $m_Q$  and  $m$ , and the strong coupling  $\alpha_s$  denote the  $\overline{\text{MS}}$  running quantities at the scale  $\mu$ .

### 3. Extraction of the decay constant

In order to determine the heavy-meson decay constant  $f_Q$  from the OPE, we must execute the following two steps.

#### 3.1. The Borel window

First, we must fix our working  $\tau$ -window where, on the one hand, the OPE gives a sufficiently accurate description of the exact correlator (i.e., all higher-order radiative and power corrections are small) and, on the other hand, the ground state gives a “sizeable” contribution to the correlator. Since the radiative corrections to the condensates increase rather fast with  $\tau$ , it is preferable to stay at the lowest possible values of  $\tau$ . We shall therefore fix the window by the following criteria [7,8]: (a) In the window, power corrections  $\Pi_{\text{power}}(\tau)$  should not exceed 30% of the dual correlator  $\Pi_{\text{dual}}(\tau, s_0)$ . This restricts the upper boundary of the  $\tau$ -window. The ground-state contribution to the correlator at this value of  $\tau$  comprises about 50% of the correlator. (b) The lower boundary of the  $\tau$ -window is fixed by the requirement that the ground-state contribution does not fall below 10%.

#### 3.2. The effective continuum threshold

Second, we must define a criterion how to determine  $s_{\text{eff}}(\tau)$ . The corresponding algorithm has been formulated in our recent works [7,8] and was shown to provide a good extraction of the ground-state parameters in quantum-mechanical potential models.

Let us introduce the *dual invariant mass*  $M_{\text{dual}}$  and the *dual decay constant*  $f_{\text{dual}}$  by the relations

$$M_{\text{dual}}^2(\tau) \equiv -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)),$$

$$f_{\text{dual}}^2(\tau) \equiv M_Q^{-4} e^{M_Q^2 \tau} \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)). \quad (3.1)$$

For a properly constructed  $\Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau))$ , this dual mass should coincide with the actual mass of the ground state. So, if the ground-state mass is known, any deviation of the dual mass from the actual mass of the ground state yields an indication of the contamination of the dual correlator by excited states.

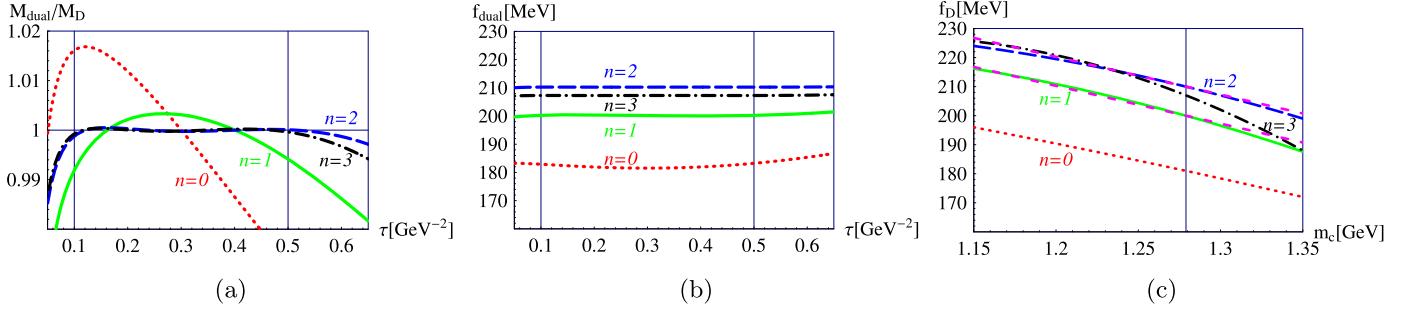
Assuming some particular functional form of the effective threshold and requiring the least deviation of the dual mass (3.1) from the actual mass in the  $\tau$ -window entails a variational solution for the effective threshold; as soon as the latter has been fixed, (3.1) yields the decay constant. We do not need to describe the effective threshold in the whole range of values of  $\tau$ ; it is sufficient to parameterize the  $\tau$ -dependence in the working Borel window where a local Taylor-expansion may be assumed. We therefore consider polynomials in  $\tau$ , including also the standard assumption for the effective threshold – a  $\tau$ -independent constant.

Our algorithm for the extraction of  $f_Q$  makes use of the knowledge of the true  $P_Q$ -meson mass  $M_Q$ . This algorithm, developed in our previous works and proven to work well for different correlators in the potential model, is very simple: we consider the set of  $\tau$ -dependent Ansätze for the effective continuum threshold

$$s_{\text{eff}}^{(n)}(\tau) = \sum_{j=0}^n s_j^{(n)} \tau^j. \quad (3.2)$$

We fix the parameters on the right-hand side of (3.2) as follows: we compute the dual mass squared according to (3.1) for the  $\tau$ -dependent  $s_{\text{eff}}(\tau)$  in (3.2). We then evaluate  $M_{\text{dual}}^2(\tau)$  at several values of  $\tau = \tau_i$  ( $i = 1, 2, \dots, N$ , where  $N$  can be taken arbitrary large) chosen uniformly in the window. Finally, we minimize the squared difference between  $M_{\text{dual}}^2$  and the known value  $M_B^2$ :

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}^2(\tau_i) - M_Q^2]^2. \quad (3.3)$$



**Fig. 2.** Dual mass (a) and dual decay constant (b) of the  $D$  meson obtained using different Ansätze for the effective continuum threshold  $s_{\text{eff}}(\tau)$  (3.2) and fixing all thresholds according to (3.3). Results for  $m_c \equiv \bar{m}_c(\bar{m}_c) = 1.279$  GeV,  $\mu = m_c$ , and central values of the other relevant parameters are presented. (c) Dual decay constant of the  $D$  meson vs.  $m_c$  for  $\mu = m_c$  and central values of the other OPE parameters. The integer  $n = 0, 1, 2, 3$  is the degree of the polynomial in our Ansatz (3.2) for  $s_{\text{eff}}(\tau)$ : dotted line (red) –  $n = 0$ ; solid line (green) –  $n = 1$ ; dashed line (blue) –  $n = 2$ ; dash-dotted line (black) –  $n = 3$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

This gives us the coefficients  $s_j^{(n)}$  of the effective continuum threshold. As soon as the latter is fixed, it is straightforward to calculate the decay constant.

The results presented below indicate that accounting for the  $\tau$ -dependence of the effective threshold yields a visible improvement compared with the usual assumption of a  $\tau$ -independent quantity in the following respect: it leads to a much better stability of the dual mass calculated for a dual correlator, which is tantamount to a better isolation of the ground-state contribution.

Still, by trying different Ansätze for the effective continuum threshold, one obtains different estimates for the decay constant. We discuss the interpretation of these results in connection with the systematic uncertainties of the method of sum rules.

### 3.3. Uncertainties in the extracted decay constant

Clearly, the extracted value of the decay constant is sensitive to the precise values of the OPE parameters and to the prescription for fixing the effective continuum threshold. The corresponding errors in the resulting decay constants are called the *OPE-related error* and the *systematic error*, respectively. Let us discuss these in turn.

#### 3.3.1. OPE-related error

The value of the OPE-related error is obtained as follows: We perform a bootstrap analysis [9] by allowing the OPE parameters to vary over the ranges indicated in (2.1), using 1000 bootstrap events. Gaussian distributions for all OPE parameters but  $\mu$  are employed. For  $\mu$  we assume a uniform distribution in the corresponding range, which we choose to be  $1 \leq \mu(\text{GeV}) \leq 3$  for charmed mesons and  $2 \leq \mu(\text{GeV}) \leq 8$  for beauty mesons. The resulting distribution of the decay constant turns out to be close to Gaussian shape. Therefore, the quoted OPE-related error is a Gaussian error.

#### 3.3.2. Systematic error

The systematic error of any hadron parameter determined by the method of sum rules (i.e., the error related to the intrinsic limited accuracy of this method) represents the perhaps most subtle point in the applications of this method. So far no way to arrive at a *rigorous* – in the mathematical sense – systematic error has been proposed. Therefore, in this respect we have to rely on our experience obtained from the examples where the exact hadron parameters may be calculated independently from the method of dispersive sum rules and then compared with the results of the sum-rule approach. Working with polynomial parameterizations in the case of potential models, we have seen that the band of values obtained from linear, quadratic, and cubic Ansätze for the effective threshold encompasses the true value of the decay constant

[7]. Moreover, we could show that the extraction procedures in quantum mechanics and in QCD are even quantitatively rather similar [8]. Therefore, we believe that the half-width of this band may be regarded as *realistic* estimate for the systematic uncertainty of the decay constant. Presently, we do not see other possibilities to obtain a more reliable estimate for the systematic error.

### 3.4. Decay constant of the $D$ meson

The  $\tau$ -window for the charmed mesons,  $\tau = (0.1\text{--}0.5)$  GeV $^{-2}$ , is chosen according to the criteria formulated above. Fig. 2 shows the application of our procedure of fixing the effective continuum threshold and extracting the resulting  $f_D$ . We would like to point out that, in the window, the  $\tau$ -dependent effective thresholds reproduce the meson mass much better than the constant one (Fig. 2(a)). This signals that the dual correlators corresponding to the  $\tau$ -dependent thresholds are less contaminated by excited states.

The dependence of the extracted value of the  $D$ -meson decay constant  $f_D$  on the  $c$ -quark mass  $m_c \equiv \bar{m}_c(\bar{m}_c)$  and the condensate  $\langle \bar{q}q \rangle \equiv \langle \bar{q}q(2 \text{ GeV}) \rangle$  may be parameterized as

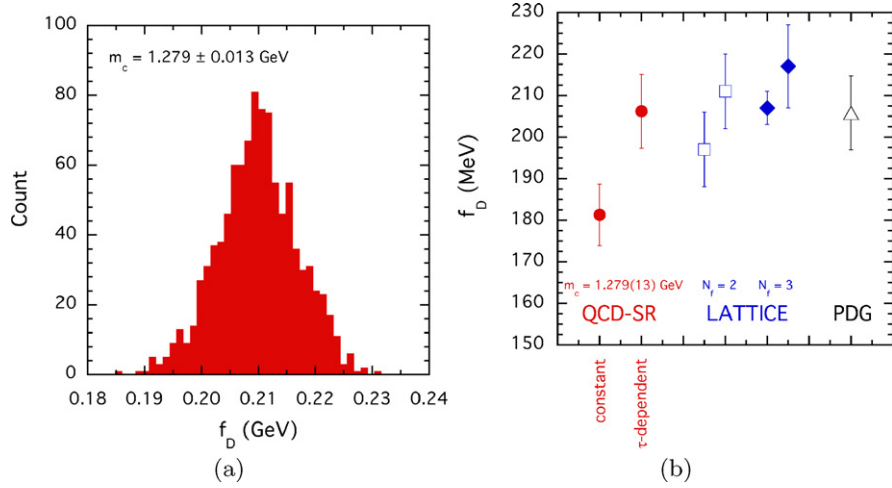
$$f_D^{\text{dual}}(m_c, \mu = m_c, \langle \bar{q}q \rangle) = \left[ 206.2 - 13 \left( \frac{m_c - 1.279 \text{ GeV}}{0.1 \text{ GeV}} \right) + 4 \left( \frac{|\langle \bar{q}q \rangle|^{1/3} - 0.267 \text{ GeV}}{0.01 \text{ GeV}} \right) \pm 5.1_{(\text{sys})} \right] \text{ MeV}. \quad (3.4)$$

This formula describes the band of values indicated by the two short-dashed lines in Fig. 2(c), which delimit the results found from the linear, quadratic, and cubic Ansätze for the effective continuum threshold. Fig. 3(a) depicts the result of the bootstrap analysis of the OPE uncertainties. The distribution has a Gaussian shape, and therefore the corresponding OPE uncertainty is the Gaussian error. Adding the half-width of the band deduced from our  $\tau$ -dependent Ansätze for the effective continuum threshold of degree  $n = 1, 2, 3$  as the (intrinsic) systematic error, we obtain the following result:

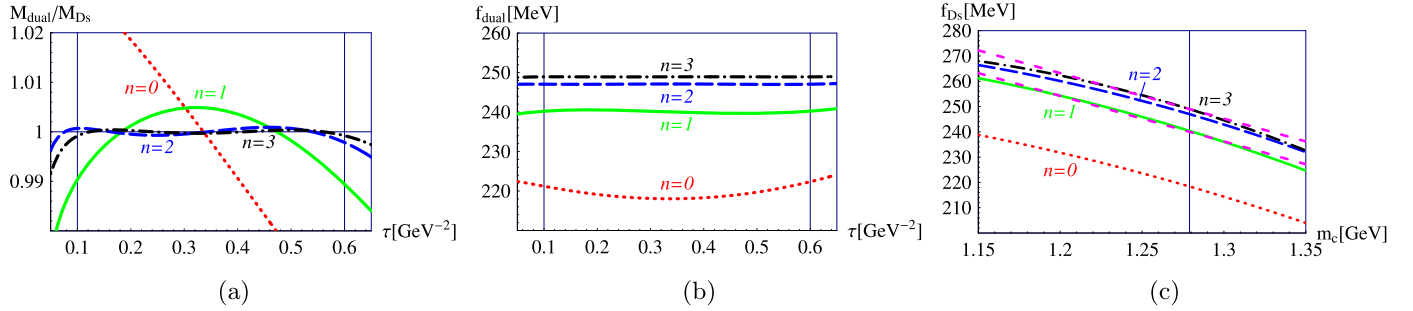
$$f_D = (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{sys})}) \text{ MeV}. \quad (3.5)$$

The main sources of the OPE uncertainty in the extracted  $f_D$  are its renormalization-scale dependence and the error of the quark condensate.

For a  $\tau$ -independent Ansatz for the effective continuum threshold a bootstrap analysis entails the substantially lower range  $f_D^{(n=0)} = (181.3 \pm 7.4_{(\text{OPE})}) \text{ MeV}$ , which differs from our  $\tau$ -dependent result (3.5) by  $\simeq 10\%$ , i.e., by almost three times the OPE



**Fig. 3.** (a) Distribution of  $f_D$  obtained by the bootstrap analysis of the OPE uncertainties. Gaussian distributions for all OPE parameters but  $\mu$  with corresponding errors as given in (2.1) are employed. For  $\mu$  we assume a uniform distribution in the range  $1 \text{ GeV} < \mu < 3 \text{ GeV}$ . (b) Summary of findings for  $f_D$ . Lattice results are from [10,11] for two dynamical light flavors ( $N_f = 2$ ) and from [12,13] for three dynamical flavors ( $N_f = 3$ ). The triangle represents the experimental value from PDG [14]. The estimate obtained with the constant threshold includes the OPE uncertainty only; for the  $\tau$ -dependent QCD-SR result the error shown is the sum of the OPE and systematic uncertainties in (3.5), added in quadrature.



**Fig. 4.** Same as Fig. 2 but for the  $D_s$  meson.

uncertainty. Moreover, as we have already shown in our previous works [6], making use of merely the constant Ansatz for the effective continuum threshold does not allow one to probe at all the intrinsic systematic error of the QCD sum rule. From our result (3.5) the latter turns out to be of the same order as the OPE uncertainty.

Allowing the threshold to depend on  $\tau$  leads to a clearly visible effect and brings the results from QCD sum rules into perfect agreement with recent lattice results and the experimental data (Fig. 3(b)). This perfect agreement of our result with both experimental data and lattice results provides a strong argument in favour of the reliability of our procedure.

### 3.5. Decay constant of $D_s$ meson

The corresponding  $\tau$ -window is  $\tau = (0.1-0.6) \text{ GeV}^{-2}$ . Fig. 4 provides the details of our extraction procedure. Our results for the  $D_s$ -meson decay constant  $f_{D_s}$  may be represented as

$$f_{D_s}^{\text{dual}}(m_c, \mu = m_c, \langle \bar{s}s \rangle) = \left[ 245.3 - 18 \left( \frac{m_c - 1.279 \text{ GeV}}{0.1 \text{ GeV}} \right) + 3.5 \left( \frac{|\langle \bar{s}s \rangle|^{1/3} - 0.248 \text{ GeV}}{0.01 \text{ GeV}} \right) \pm 4.5_{(\text{syst})} \right] \text{ MeV.} \quad (3.6)$$

This formula describes the band of values indicated by the two short-dashed lines in Fig. 4(c) as function of  $m_c \equiv \bar{m}_c(\bar{m}_c)$  and

gives also the dependence on the quark condensate  $\langle \bar{s}s \rangle \equiv \langle \bar{s}s(2 \text{ GeV}) \rangle$ . Performing the bootstrap analysis of the OPE uncertainties, we obtain the following estimate:

$$f_{D_s} = (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{syst})}) \text{ MeV.} \quad (3.7)$$

As in the case of  $f_D$ , a constant-threshold Ansatz yields a substantially lower value:  $f_{D_s}^{(n=0)} = (218.8 \pm 16.1_{(\text{OPE})}) \text{ MeV}$ . (See Fig. 5.)

### 3.6. $f_{D_s}/f_D$

For the ratio of the  $D$  and  $D_s$  decay constants we report the sum-rule prediction

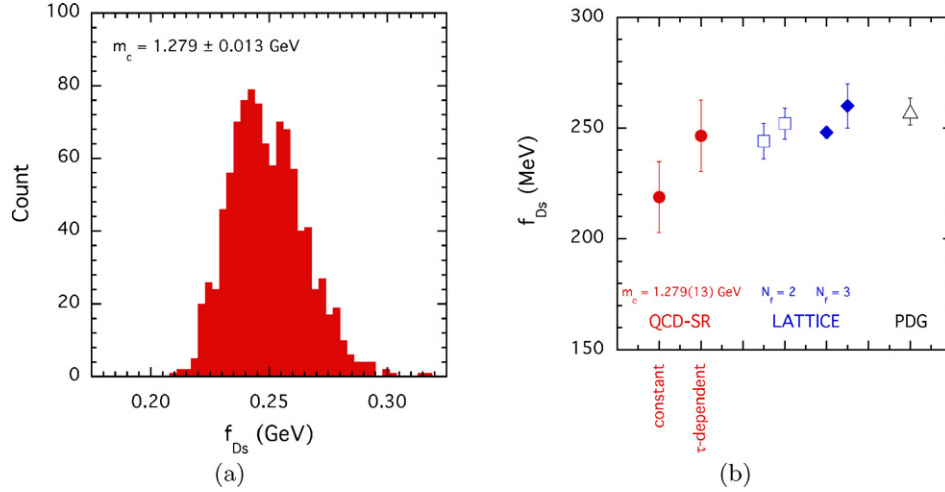
$$f_{D_s}/f_D = 1.193 \pm 0.025_{(\text{OPE})} \pm 0.007_{(\text{syst})}. \quad (3.8)$$

This value is to be compared with the PDG average  $f_{D_s}/f_D = 1.25 \pm 0.06$  [14] as well as with the recent lattice results  $f_{D_s}/f_D = 1.24 \pm 0.03$  [10] for  $N_f = 2$  and  $f_{D_s}/f_D = 1.164 \pm 0.011$  [12] and  $f_{D_s}/f_D = 1.20 \pm 0.02$  [13] for  $N_f = 3$ . The error in (3.8) arises mainly from the uncertainties in the quark condensates  $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle = 0.8 \pm 0.3$ .

## 4. Summary and conclusions

We presented a detailed analysis of the decay constants of charmed heavy mesons with the help of QCD sum rules. Particular emphasis was laid on the study of the uncertainties in the





**Fig. 5.** (a) Distribution of  $f_{D_s}$  obtained by the bootstrap analysis of the OPE uncertainties. Gaussian distributions for all OPE parameters but  $\mu$  with corresponding errors as given in (2.1) are employed. For  $\mu$  we assume a uniform distribution in the range  $1 \text{ GeV} < \mu < 3 \text{ GeV}$ . (b) Summary of findings for  $f_{D_s}$ . Lattice results are from [10,11] for two dynamical light flavors ( $N_f = 2$ ) and from [12,13] for three dynamical flavors ( $N_f = 3$ ). The triangle represents the experimental value from PDG [14]. The estimate obtained with the constant threshold includes the OPE uncertainty only; for the  $\tau$ -dependent QCD-SR result the error shown is the sum of the OPE and systematic uncertainties in (3.7), added in quadrature.

extracted values of the decay constants: the OPE uncertainty related to the not precisely known QCD parameters and the intrinsic uncertainty of the sum-rule method related to a limited accuracy of the extraction procedure.

Our main findings may be summarized as follows.

- (i) The perturbative expansion of the two-point function in terms of the pole mass of the heavy quark exhibits no sign of convergence. However, reorganizing this expansion in terms of the corresponding running mass leads to a clear hierarchy of the perturbative contributions. Interestingly, the decay constant extracted from the pole-mass OPE proves to be sizeably smaller than the one extracted from the running-mass OPE. In spite of this numerical difference, the decay constants extracted from these two correlators exhibit perfect stability in the Borel parameter. This example shows that stability *per se* does not guarantee the reliability of the sum-rule extraction of any bound-state parameter.
- (ii) We have made use of the Borel-parameter-dependent effective threshold for the extraction of the decay constants. The  $\tau$ -dependence of the effective threshold emerges quite naturally when one attempts to increase the accuracy of the duality approximation. According to our algorithm, one should consider different polynomial Ansätze for the effective threshold and fix the coefficients in these Ansätze by minimizing the deviation of the dual mass from the known actual meson mass in the window. Then, the band of values corresponding to the linear, quadratic, and cubic Ansätze reflects the intrinsic uncertainty of the method of sum rules. The efficiency of this criterion has been tested before for several examples of quantum-mechanical models. This strategy has now been applied to the decay constants of heavy mesons.
- (iii) We obtained the following sum-rule estimates for the decay constants of the charmed  $D$  and  $D_s$  mesons:

$$f_D = (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{syst})}) \text{ MeV}, \quad (4.1)$$

$$f_{D_s} = (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{syst})}) \text{ MeV}. \quad (4.2)$$

We point out that we provide both the OPE uncertainties and the intrinsic (systematic) uncertainty of the method of sum rules related to the limited accuracy of the extraction procedure. In the case of  $f_D$  the latter turns out to be of the

same order as the OPE uncertainty. Noteworthy, adopting a  $\tau$ -independent effective threshold leads to a substantially lower range  $f_D^{(n=0)} = (181.3 \pm 7.4_{(\text{OPE})}) \text{ MeV}$ , which differs from our  $\tau$ -dependent result (4.1) by almost three times the OPE uncertainty. The resulting ratio of the decay constants is

$$f_{D_s}/f_D = 1.193 \pm 0.025_{(\text{OPE})} \pm 0.007_{(\text{syst})}. \quad (4.3)$$

- (iv) Our study of charmed mesons clearly demonstrates that the use of Borel-parameter-dependent thresholds leads to two essential improvements:
  - a. The actual accuracy of the decay constants extracted from sum rules improves considerably.
  - b. Our algorithm yields *realistic* (although not entirely rigorous) estimates for the systematic errors and allows one to reduce their values to the level of a few percent. Due to the application of our prescription, the QCD sum-rule results are brought into perfect agreement both with the experimental results and with lattice QCD.

## Acknowledgements

D.M. was supported by the Austrian Science Fund (FWF), projects Nos. P20573 and P22843. The work was supported in part by a Grant for Leading Scientific Schools 1456.2008.2, and by FASI state Contract No. 02.740.11.0244.

## References

- [1] M. Shifman, A. Vainshtein, V. Zakharov, Nucl. Phys. B 147 (1979) 385.
- [2] T.M. Aliev, V.L. Eletsky, Yad. Fiz. 38 (1983) 1537.
- [3] K.G. Chetyrkin, M. Steinhauser, Phys. Lett. B 502 (2001) 104; K.G. Chetyrkin, M. Steinhauser, Eur. Phys. J. C 21 (2001) 319.
- [4] M. Jamin, B.O. Lange, Phys. Rev. D 65 (2002) 056005.
- [5] K.G. Chetyrkin, et al., Phys. Rev. D 80 (2009) 074010.
- [6] W. Lucha, D. Melikhov, S. Simula, Phys. Rev. D 76 (2007) 036002; W. Lucha, D. Melikhov, S. Simula, Phys. Lett. B 657 (2007) 148; W. Lucha, D. Melikhov, S. Simula, Phys. Atom. Nucl. 71 (2008) 1461; W. Lucha, D. Melikhov, S. Simula, Phys. Lett. B 671 (2009) 445; D. Melikhov, Phys. Lett. B 671 (2009) 450.
- [7] W. Lucha, D. Melikhov, S. Simula, Phys. Rev. D 79 (2009) 096011; W. Lucha, D. Melikhov, S. Simula, J. Phys. G 37 (2010) 035003; W. Lucha, D. Melikhov, H. Sazdjian, S. Simula, Phys. Rev. D 80 (2009) 114028.
- [8] W. Lucha, D. Melikhov, S. Simula, Phys. Lett. B 687 (2010) 48; W. Lucha, D. Melikhov, S. Simula, Phys. Atom. Nucl. 73 (2010) 1770.

- [9] B. Efron, R.J. Tibshirani, An Introduction to the Bootstrap, Monographs on Statistics and Applied Probability, vol. 57, CRC Press, 1993.
- [10] B. Blossier, et al., ETM Collaboration, JHEP 0907 (2009) 043.
- [11] B. Blossier, et al., ETM Collaboration, JHEP 1004 (2010) 049.
- [12] E. Follana, C.T.H. Davies, G.P. Lepage, J. Shigemitsu, HPQCD Collaboration, UKQCD Collaboration, Phys. Rev. Lett. 100 (2008) 062002; C.T.H. Davies, C. McNeile, E. Follana, G.P. Lepage, H. Na, J. Shigemitsu, HPQCD Collaboration, Phys. Rev. D 82 (2010) 114504.
- [13] A. Bazavov, et al., Fermilab Lattice Collaboration, MILC Collaboration, PoS LAT2009 (2009) 249.
- [14] K. Nakamura, et al., Particle Data Group, J. Phys. G 37 (2010) 075021.